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On the Radius of Unicity for Nonlinear Families

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INTRODUCTION

We shall consider an approximation problem of the following type. An open set $S \subset E^N$ and a map $A : S \to H$ are given where H is some real inner product space. Given $f \in H$ the approximation problem is to minimize $\| A(x) - f \|^2 = [A(x) - f, A(x) - f]$ as x ranges over S. Moreover, we assume that the map A is twice continuously Fréchet differentiable.

In particular we shall obtain results of the sort: "If $x_0 \in S$ and $\| A(x_0) - f \|$ is sufficiently small then f has a unique (global) best approximation in $A(S)$." Moreover, sufficiently small will be defined by an explicit bound, the calculation of which involves x_0 only. As a result we will also be able to obtain a lower bound on what might be called the radius of unicity at x_0. This would be defined to be the supremum of the set of $r \geqslant 0$ such that $\| A(x_0) - f \| < r$ implies that f has a unique best approximation in $A(S)$. That is, given $x_0 \in S$ such that x_0 is normal (defined later) it will be possible for certain types of nonlinear families to explicitly determine δ_0 such that if $\| f - A(x_0) \| < \delta_0$ then f has a unique best approximation in $A(S)$.

Results of the first type mentioned above (under the assumption that x_0 is a local best approximation) may be found in [1, 2, 3]. Several numerical examples are given in [2]. The calculation of the type of bounds given in these papers requires, in general, the minimization of certain nonlinear functions on S. The type of bounds given in this paper seem simpler and more explicit.

BASIC RESULTS

Let $A : S \subset E^N \to H$ be such that the map $x \to A''(x, \cdot, \cdot)$ is continuous on S and let $f \in H$ be arbitrary but fixed. For $x_0 \in S$ let S_0 denote the level set $\{x \in S \mid \| A(x) - A(x_0) \| \leqslant 2 \| A(x_0) - f \|\}$. Finally for $x \in S$ let $\psi(x) = ([A(x) - f, (\partial A/\partial x_1)(x)], \ldots, [A(x) - f, (\partial A/\partial x_N)(x)])^T$ and note that since $2\psi(x)$ is the gradient of $F(x) = [A(x) - f, A(x) - f]$ a necessary condition that $\bar{x}$ be a local minimum of $F(x)$ is that $\psi(\bar{x}) = 0$.

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Assume that $x_0 \in S$ is such that there exist positive constants K_0, K_1, and γ depending only on x_0 and $\| A(x_0) - f\|$ such that

(i) $\| A(x) - A(x_0)\| \geqslant \gamma \| x - x_0 \|$,

(ii) $\| \psi'(x) - \psi'(x_0)\| \leqslant K_0 \| x - x_0 \| + K_1 \| A(x_0) - f \|$,

for all $x \in S_0$ where the usual Euclidean norm is used on E^N.

We shall need the following result a proof of which may be found in [4, p. 200].

LEMMA 1. *Let $f: X \to Y$, X and Y real Banach spaces have a continuous derivative for x in some open set U. Let $x_0 \in U$ be such that $(f'(x_0))^{-1} = f'_{-1}(x_0)$ exists. For some L, $0 < L < 1$, let r be such that the inequality*

(i) $\| f'_{-1}(x_0)\| \| f'(x) - f'(x_0)\| < L$ *holds for all x such that $\| x - x_0 \| \leqslant r$.*

Suppose that

(ii) $\| f'_{-1}(x_0) f(x_0)\| < r(1 - L)$.

Then there exists a unique zero x of f in $B(x_0, r) = \{x \mid \| x - x_0 \| \leqslant r\}$ and the sequence $\{x_{n+1}\}$ defined by $x_{m+1} = x_m - f'_{-1}(x_0) f(x_m)$ converges to x^* and satisfies $\| x^* - x_m \| \leqslant (L^m/(1 - L))\| x_1 - x_0 \|$.

An examination of the proof given in [4] shows that X may be replaced by an open subset V of X provided that $\bar{B}(x_0, r) \subset V$. This extended form is the version we shall use.

THEOREM 1. *In the above setting assume that $\epsilon_0 = \| A(x_0) - f \|$ satisfies*

(1) $\epsilon_0 \leqslant \lambda_0/2K_2$,

(2) $\epsilon_0 < \lambda_0/4(K_1 + K_0K_4)$,

(3) $B(x_0; r) = \{x \mid \| x - x_0 \| \leqslant r\} \subset S$,

when $\epsilon_0 K_4 \leqslant r \leqslant \lambda_0/4K_0 - \epsilon_0 K_1/K_0$ where $K_2 = \| A''(x_0, \cdot, \cdot)\|$, $\lambda_0 = \inf_{\|h\|=1} [A'(x_0, h), A'(x_0, h)]$, $K_3 = [\sum_{i=1}^{n} \|(\partial A/\partial x_i)(x_0)\|^2]^{1/2}$ and $K_4 = \max\{2/\gamma, 4K_3/\lambda_0\}$.

Then there is a unique solution x^* of $\psi(x) = 0$ lying in the ball $B(x_0; r)$. Moreover, the sequence $X_{\nu+1} = X_\nu - \psi'_{-1}(x_0) \psi(x_\nu)$ converges to x^* with $\| x_\nu - x^* \| \leqslant r/2^{\nu-1}$ $\nu = 1, 2, \ldots$.

Proof. We shall check that the hypotheses of Lemma 1 are satisfied at x_0 with $L = \frac{1}{2}$. First note that $\epsilon_0 < (\lambda_0/4(K_1 + K_0K_4))$ if and only if $\epsilon_0 K_4 < (\lambda_0/4K_0) - (K_1/K_0) \epsilon_0$ so that positive r satisfying the inequality in (3) exists. Inequality (1) implies that the quadratic form $\psi'(x_0, h, k) = [A'(x_0, h), A'(x_0, k)] + [A(x_0) - f, A''(x_0, h, k)]$ is positive definite with minimum eigenvalue at least $\lambda_0/2$ since $\inf_{\|h\|=1} \psi'(x_0, h, h) \geqslant \lambda_0 - \| A(x_0) - f \|$

$\| A''(x_0, \cdot, \cdot)\| \geqslant \lambda_0 - \epsilon_0 K_2 \geqslant \lambda_0/2$ by (1). Thus, $\| \psi'_{-1}(x_0)\| \leqslant 2/\lambda_0$ and letting $L = \frac{1}{2}$ in Lemma 1 we have that for $\| x - x_0 \| \leqslant r$

$$\text{(a)}\quad \| \psi'_{-1}(x_0)\| \, \| \psi'(x) - \psi'(x_0)\| \leqslant \frac{2}{\lambda_0} [K_0 \| x - x_0 \| + K_1 \epsilon_0]$$

$$\leqslant \frac{2K_0 r}{\lambda_0} + \frac{2K_1\epsilon_0}{\lambda_0} < L = \tfrac{1}{2}$$

since $r < (\lambda_0/4K_0) - (\epsilon_0 K_1/K_0)$ implies that $r + (K_1\epsilon_0/K_0) < (\lambda_0/4K_0)$ so that $K_0 r + K_1\epsilon_0 < \lambda_0/4$ yielding $2/\lambda_0[K_0 r + K_1\epsilon_0] < \frac{1}{2} = L$.

$$\text{(b)}\quad \| \psi'_{-1}(x_0)\, \psi(x_0)\| \leqslant \frac{2}{\lambda_0} \| \psi(x_0)\| \leqslant \frac{2}{\lambda_0}\epsilon_0 K_3 < r/2 = r(1 - L).$$

since $r > \epsilon_0 K_4 \geqslant (4\epsilon_0 K_3/\lambda_0)$.

Hence by Lemma 1 the conclusion of the theorem is valid. ∎

Remark 1. The conclusion of Theorem 1 becomes trivial in case $\lambda_0 = 0$ for then $f = A(x_0)$ if (1) is satisfied. A point $x_0 \in S$ for which $\lambda_0 > 0$ is usually called a *normal* point. The usual situation encountered is that any solution to $\psi(x) = 0$ must be normal and this is true in particular for the rational family we consider later in this paper [3].

COROLLARY. *Suppose $x_0 \in S$ be such that the hypotheses of Theorem* 1 *are satisfied. Then each $f \in H$ in the open ball of radius δ_0 about $A(x_0)$ has a unique global best approximation in $A(S)$ where*

$$\delta_0 = \lambda_0 \min \left\{ \frac{1}{2K_2}, \frac{1}{4(K_1 + K_0 K_4)} \right\}.$$

Proof. Let f be arbitrary but fixed with $\| A(x_0) - f \| < r$ where r is any number satisfying (3) in Theorem 1. If $x \in S_0$ then $\| x - x_0 \| \leqslant (1/\gamma)\| A(x) - A(x_0)\| \leqslant 2/\gamma \| A(x_0) - f \| \leqslant \epsilon_0 K_4 < r$. Thus, $S_0 \subset B(x_0; r)$ and so S_0 is compact. Hence $F(x) \equiv [A(x) - f, A(x) - f]$ achieves a minimum (over S_0) at $x^* \in S_0$ which is clearly also a minimum over S. Thus $\psi(x^*) = 0$ and by Theorem 1, x^* is unique. Since f was arbitrary the result follows. ∎

Remark 2. If we assume that x_0 is itself a local minimum of $[A(x) - f, A(x) - f]$ then Theorem 1 in this case may be interpreted as a test for determining whether x_0 is actually a global minimum. Results of this type are considered in [1, 2, 3]. The bounds on ϵ_0 given in these papers require the minimization of certain nonlinear functions over the entire set S_0 rather than the evaluation of quantities directly calculable in terms of x_0 itself. We have no information, however, on how the bounds of this paper compare in size to those given in [1] or [2].

APPLICATIONS

We now consider two applications of Theorem 1 to specific approximating families. The first of these is a class of families discussed in [1, 3]. The second is the family of rational functions having only real poles.

EXAMPLE 1. Let T be a compact Hausdorff space and m a regular Borel measure on T. Suppose $\{v_1, ..., v_n\}$ is an independent subset of $C(T)$ (the real-valued continuous functions on T) with the property that each nonzero g in span$\{v_1, ..., v_n\}$ is such that $m\{t \mid g(t) = 0\} = 0$. Let $f: E^1 \to E^1$ be thrice differentiable and satisfy $M \geqslant f'(s) \geqslant \alpha > 0$, $|f''(s)| \leqslant \rho$ and $|f'''(s)| \leqslant C$ for all $s \in E^1$ (e.g., $f(s) = s + \arctan(s)$). Define $A: E^n \to L_2(T, M)$ by $A(x)(t) = f(\sum_{i=1}^n x_i v_i(t))$. Note that $N(x) = \| \sum_{i=1}^n x_i v_i(t) \|$ defines a norm on E^n. For convenience we assume that $m(T) = 1$.

For each $t \in T, h \in E^N$, $A(x + h)(t) - A(x)(t) = f(\sum_j (x_j + h_j) v_j(t)) - f(\sum_j x_j v_j(t)) = f'(\sum_j (x_j + \theta_t h_j) v_j(t)) \sum_j h_j v_j(t)$ where $0 < \theta_t < 1$ using the mean value theorem. Thus

$$\| A(x + h) - A(x) \| = \left[\int_T f'\left(\sum_j (x_j + \theta_t h_j) v_j(t) \right)^2 \left(\sum_i h_i v_i(t) \right)^2 dm \right]^{1/2} \geqslant \alpha\beta \| h \|$$

where $\beta > 0$ such that $\| \sum_i k_i v_i(t) \| \geqslant \beta \| k \|$ for all $k \in E^n$. A similar calculation also shows that $\inf_{\|h\|=1} A'(x_0, h) \| \geqslant \alpha\beta \| h \|$.

To estimate $\| \psi'(x) - \psi'(x_0) \|$ we first recall that for a real symmetric matrix B, $\| B \| = \max_{\|h\|=1} |\langle h, Bh \rangle|$ where $\langle \cdot, \cdot \rangle$ is the usual inner product on E^N. Now, $|\langle h, \psi'(x) h \rangle - \langle h, \psi'(x_0) h \rangle| = |[A(x) - g, A''(x, h, h)] + [A'(x, h), A'(x, h)] - [A(x_0) - g, A''(x_0, h, h)] - [A'(x_0, h), A'(x_0, h)]|$ where $g \in L_2(T, M)$ is the function that is to be approximated. Applying the triangle inequality we arrive at the inequality

$$\begin{aligned} |\langle h, (\psi'(x) - \psi'(x_0)) h \rangle| \leqslant & \| A'(x_0, h) \| \| A'(x, h) - A'(x_0, h) \| \\ & + \| A'(x, h) \| \| A'(x, h) - A'(x_0, h) \| \\ & + \| A''(x, h, h) \| \| A(x) - A(x_0) \| \\ & + \| A(x_0) - g \| \cdot \| A''(x, h, h) - A''(x_0, h, h) \|. \quad (1) \end{aligned}$$

To estimate the right-hand side of (1), let σ denote the quantity $\| \sum_{j=1}^n v_j^2(t) \|_\infty^{1/2}$. Then some elementary but tedious calculations yield for $\| h \| = 1$

(i) $\| A'(x, h) \| \leqslant M\sigma$,

(ii) $\| A'(x, h) - A'(x_0, h) \| \leqslant \rho\sigma^2 \| x - x_0 \|$,

(iii) $\| A''(x, h, h) \| \leqslant \rho\sigma^2$,

(iv) $\| A(x) - A(x_0) \| \leqslant M\sigma \| x - x_0 \|$,

(v) $\| A''(x, h, h) - A''(x_0, h, h) \| \leqslant C\sigma^3 \| x - x_0 \|$.

Using the above and (1) we find that in Theorem 1 we may take $K_0 = \sigma^3(\rho M + C\epsilon_0)$, $K_1 = 0$, that $K_2 = \rho\sigma^2$, $\lambda_0 \geqslant \alpha\beta$, $K_3 \leqslant M\sigma$, $\gamma \geqslant \alpha\beta$, and $K_4 \leqslant \max\{2/\alpha\beta, 4K_3/\alpha\beta\} \leqslant 1/\alpha\beta \max\{2, 4M\sigma\}$. Thus we have

THEOREM 2. *In the setting above, if $x_0 \in S$ satisfies*

(1) $\epsilon_0 \equiv \| A(x_0) - g \| \leqslant \alpha\beta/2\rho\sigma^2$

and

(2) $\epsilon_0 < \dfrac{\alpha^2\beta^2}{8} \cdot K_0^{-1} \max\{1, 2M\sigma\}^{-1}$

then the conclusions of Theorem 1 *and Corollary* 1 *hold. In particular, each g such that $\| A(x_0) - g \| \leqslant \epsilon_0$ has a unique best approximation in $A(S)$.*

We now consider a second application of Theorem 1.

EXAMPLE 2. Let $H = L_2[-1, 1]$ and $\mathscr{T}_{n,m} = \{P/Q \mid P(t) = a_0 + a_1t + \cdots + a_nt^n, Q(t) = 1 + b_1t + \cdots + b_mt^m$ and $Q(t) > 0$ for all $t \in [-1, 1]$ and Q has m real roots$\}$. Let $S = \{(a_0, ..., a_n, b_1, ..., b_m) | A(x) \equiv P(A)/Q(B) \equiv (a_0 + \cdots + a_nt^n)/(1 + b_1t + \cdots + b_mt^m) \in \mathscr{T}_{n,m}\}$. For each $x \in S$ the tangent space T_x at x is defined to be the linear span of

$$\left\{ \frac{1}{Q}, \frac{t}{Q}, ..., \frac{t^n}{Q}, \frac{tP}{Q^2}, ..., \frac{t^mP}{Q^2} \right\}.$$

The point $x \in S$ is called normal if $\dim T_x = m + n + 1$. In this case $\inf_{\|h\|=1} \| A'(x, h)\|^2 \equiv \lambda_0 > 0$ and is the smallest eigenvalue of the positive definite matrix

$$\left(\left[\frac{\partial A}{\partial x_i}(x), \frac{\partial A}{\partial x_j}(x)\right]\right) \quad 1 \leqslant i, j \leqslant m + n + 1.$$

Thus, $\| A'(x, h)\| \geqslant (\lambda_0)^{1/2} \| h \|$ if x is normal.

A simple calculation shows that if $x, x_0 \in S$ then

(i) $A(x) - A(x_0) = \dfrac{Q_0}{Q} \left\{ \dfrac{Q_0P - P_0Q}{Q_0^2} \right\}$,

(ii) $A'(x_0, x - x_0) = \dfrac{Q_0P - P_0Q}{Q_0^2}$,

where $A(x_0) = P_0/Q_0$ and $A(x) = P/Q$.

LEMMA 2. *For $x \in S$ the inequality $\| Q(B)\|_\infty \leqslant 2^m$ always obtains.*

Proof. For $x = (A, B) \in S$ we have that $Q(B)$ has the form $Q(B)(t) = \prod_{i=1}^m (1 - Z_it)$ where $Z_i \in (-1, 1)$ since $Q(B)$ does not vanish on $[-1, 1]$ and has only real roots. Thus $| Q(B)(t)| \leqslant \prod_{i=1}^m (1 + | Z_it |) \leqslant \prod_{i=1}^m 2 = 2^m$. ∎

LEMMA 3. *Suppose $x_0 \in S$ is normal and let $x \in S$ be such that $\| A(x) - A(x_0)\| \leqslant 2 \| A(x_0) - f \|$. Then*

$$\| A(x) - A(x_0)\| \geqslant \frac{\delta_0(\lambda_0)^{1/2}}{2^m} \| x - x_0 \| \qquad \textit{where } \lambda_0 = \inf_{\|h\|=1} \| A'(x_0, h)\|,$$

$\delta_0 = \inf_{t\in[-1,1]} | Q_0(t)|$, *and* $\| x - x_0 \|$ *is the usual Euclidean norm on* E^{m+n+1}.

Proof. By (i) and (ii) above we have that $A(x) - A(x_0) = (Q_0/Q) A'(x_0, x - x_0)$. Thus,

$$\| A(x) - A(x_0)\|^2 = \|(Q_0/Q) A'(x_0, x - x_0)\|^2 = \int_{-1}^{1} \left(\frac{Q_0}{Q}\right)^2 \left(\frac{Q_0 P - P_0 Q}{Q_0^2}\right)^2 dt$$

$$\geqslant \frac{\delta_0^2}{\| Q \|_\infty^2} \| A'(x_0, x - x_0)\|^2 \geqslant \frac{\delta_0^2 \lambda_0}{2^m}$$

so that

$$\frac{\delta_0(\lambda_0)^{1/2}}{2^m} \| x - x_0 \| \leqslant \| A(x) - A(x_0)\|. \quad \blacksquare$$

To apply Theorem 1 it is again necessary to estimate $|\langle h, (\psi'(x) - \psi'(x_0)) h\rangle|$ when $\| h \| = 1$. A simple calculation using the triangle inequality yields

$$\begin{aligned}|\langle h, (\psi'(x) - \psi'(x_0)) h\rangle| \leqslant \| A'(x, h) - A'(x_0, h)\| \{2 \| A'(x_0, h)\| \\ + \| A'(x, h) - A'(x_0, h)\| \} + \| A(x) - A(x_0)\| \\ \cdot \{ \| A''(x_0, \cdot, \cdot)\| + \| A''(x, h, h) \\ - A''(x_0, h, h)\| \} + \| A(x_0) - f \| \\ \cdot \| A''(x, h, h) - A''(x_0, h, h)\| \qquad (*)\end{aligned}$$

where $f \in L_2[-1, 1]$ is the function to be approximated. The calculations needed to estimate the right-hand side of (*) are even more tedious than in Example 1 but are still quite straightforward. We will consider in detail the estimations of $\| A'(x, h) - A'(x_0, h)\|$ since the technique in the other cases is essentially the same.

As in Lemmas 2 and 3 let $\delta_0 = \inf_{-1\leqslant t\leqslant 1} Q_0(t)$ and $\lambda_0 = \inf_{\|h\|=1} \| A'(x_0, h)\|^2$. Also let $\lambda_1 = \| A'(x_0, \cdot)\|^2 = \sup_{\|h\|=1} \| A'(x_0, h)\|^2$, $\sigma_{m,n} = \max\{m, n+1\}$, and $\epsilon_0 = \| A(x_0) - f \|$. Finally let P_1 and Q_1 denote the polynomials $\sum_{i=0}^{n} h_i t^i$ and $\sum_{j=1}^{m} h_{j+n} t^j$, respectively, where $h = (h_0, ..., h_n, h_{n+1}, ..., h_{m+n})$ is arbitrary but fixed with $\| h \| = 1$.

Now

$$A'(x, h) - A'(x_0, h)$$

$$= \frac{P_1(Q - Q_0)}{Q Q_0} - \frac{Q_1(Q + Q_0)}{Q^2} \left(\frac{P_0(Q - Q_0) - Q_0(P - P_0)}{Q_0^2} \right)$$

so that

$$\| A'(x, h) - A'(x_0, h)\| \leqslant \left\| \frac{1}{QQ_0} \right\|_\infty \| P_1(Q - Q_0)\| + \left\| \frac{Q + Q_0}{Q^2} \right\|_\infty \| Q_1 A'(x_0, x - x_0)\| .$$

To estimate the above, note that

$$Q = Q_0 \left(1 - \frac{Q_0 - Q}{Q_0}\right)$$

so that

$$Q_0 Q = Q_0^2 \left(1 - \frac{Q_0 - Q}{Q_0}\right).$$

Assume that

$$\epsilon_0 < \frac{\delta_0^2(\lambda_0/m)^{1/2}}{2^{m+2}}.$$

Then

$$Q_0 Q(t) \geqslant Q_0^2(t)\left(1 - \frac{\| Q - Q_0 \|_\infty}{\delta_0}\right) \geqslant Q_0^2(t)\left(1 - \frac{m^{1/2} \| x - x_0 \|}{\delta_0}\right) \geqslant \delta_0^2 \left(1 - \frac{m^{1/2} \| x - x_0 \|}{\delta_0}\right)$$

since $(Q - Q_0)(t) = \sum_{i=1}^m (b_i - b_i^{(0)}) t^i \leqslant (\sum_{i=1}^m t^{2i})^{1/2} (\sum_{i=1}^m (b_i - b_i^{(0)})^2)^{1/2} \leqslant m^{1/2} \| x - x_0 \|$, But

$$\| x - x_0 \| \leqslant \frac{2^{m+1}}{\delta_0(\lambda_0)^{1/2}} \| A(x_0) - f \|$$

by Lemma 3 so that

$$Q_0(t)\, Q(t) \geqslant \delta_0^2 \left(1 - \frac{m^{1/2} 2^{m+1} \epsilon_0}{\delta_0^2(\lambda_0)^{1/2}}\right) \geqslant \frac{\delta_0^2}{2}$$

by our assumption about ϵ_0. Thus

$$\left\| \frac{1}{QQ_0} \right\|_\infty \leqslant \frac{1}{\inf_{t \in I} Q(t)\, Q_0(t)} \leqslant \frac{2}{\delta_0^2}.$$

A similar calculation yields $\| 1/Q^2 \|_\infty \leqslant 4/\delta_0^2$ and using Lemma 2 and the triangle inequality we get

$$\left\| \frac{Q + Q_0}{Q^2} \right\|_\infty \leqslant \frac{2^{m+3}}{\delta_0^2}.$$

Proceeding to the other terms we have

$$\|P_1(Q - Q_0)\|^2$$

$$= \int_{-1}^{1} \left(\sum_{i=0}^{n} h_i t^i \right)^2 (Q - Q_0)(t)^2\, dt \leqslant \int_{-1}^{1} \left(\sum_{i=0}^{n} h_i^2 \right) \left(\sum_{i=0}^{n} t^{2i} \right) (Q - Q_0)^2 (t)\, dt$$

$$\leqslant \int_{-1}^{1} \left(\sum_{i=0}^{n} t^{2i} \right) \sum_{j=1}^{m} (b_j - b_j^{(0)})^2 \left(\sum_{j=1}^{m} t^{2j} \right) dt \leqslant \|x - x_0\|^2 \int_{-1}^{1} t^2(1 + t^2$$

$$+ \cdots + t^{2n})(1 + t^2 + \cdots + t^{2(m-1)})\, dt \leqslant \|x - x_0\|^2\, \sigma_{m,n}^2 (2/3)$$

and so $\|P_1(Q - Q_0)\|^2 \leqslant \sigma_{m,n}(2/3)^{1/2} \cdot \|x - x_0\|$. Again a very similar calculation gives the estimate $\|Q_1 \cdot A'(x_0, x - x_0)\| \leqslant (m\lambda_1)^{1/2} \|x - x_0\|$ and combining all these we have that

$$\|A'(x, h) - A'(x_0, h)\|$$

$$\leqslant \frac{2(2/3)^{1/2}\,\sigma_{m,n} + (\lambda_1 m)^{1/2}\, 2^{m+3}}{\delta_0^2} \|x - x_0\| = C_1 \|x - x_0\|.$$

In an analogous way the other terms in (*) may be estimated. The result of these calculations is contained in the following lemma.

LEMMA 4. Let $x_0 \in S$ be normal, $\delta_0 = \inf_{|t| \leqslant 1} Q_0(t) > 0$, $\lambda_0 = \inf_{\|h\| = 1} \|A'(x_0, h)\|^2$, $\lambda_1 = \|A'(x_0, \cdot)\|^2$, $\sigma_{m,n} = \max\{m, n + 1\}$, $K_2 = \|A''(x_0, \cdot, \cdot)\|$, $\gamma = (\delta_0(\lambda_0)^{1/2}/2m)$, and $\epsilon_0 = \|A(x_0) - f\|$. Define constants C_1, C_1', C_2, C_2' by

$$C_1 = \frac{2(2/3)^{1/2}\,\sigma_{m,n}}{\delta_0^2} + \frac{2^{m-3}(\lambda_1 m)^{1/2}}{\delta_0^2}.$$

$$C_2 = \frac{8(2m/3)^{1/2}\,\sigma_{m,n}}{\delta_0^3} + \frac{2^{m+5} m(\lambda_1)^{1/2}}{\delta_0^2},$$

$$C_1' = C_1\{2(\lambda_1)^{1/2} + 2C_1\epsilon_0/\gamma\},$$

and

$$C_2' = 2(\lambda_1)^{1/2} K_2 + 4(\lambda_1)^{1/2} C_2 + \epsilon_0 C_2.$$

Then for any $x \in S_0$ we have that $\|\psi'(x) - \psi'(x_0)\| \leqslant (C_1' + C_2')\|x - x_0\|$ provided that

$$\epsilon_0 < \frac{\delta_0^2(\lambda_0)^{1/2}}{m^{1/2} 2^{m+2}}.$$

This leads immediately to the following result.

THEOREM 3. *In the setting of Lemma* 4, *let* $N = m + n + 1$, $K_0 = C_1' + C_2'$,

$$K_3 = \left(\sum_{i=1}^{N} \left\| \frac{\partial A}{\partial x_i} (x_j) \right\|^2 \right)^{1/2},$$

$K_4 = \max\{2/\gamma, 4K_3/\lambda_0\}$, *and* $d_0 = \operatorname{dist}(x_0, S^c)$. *Then if*

$$\epsilon_0 < \min \left\{ \frac{d_0}{2K_4}, \frac{\lambda_0}{2K_2}, \frac{\lambda_0}{4K_0K_4}, \frac{\delta_0{}^2(\lambda_0)^{1/2}}{m^{1/2}2^{m+2}} \right\}$$

f has a unique best approximation in $\mathscr{T}_{n,m}$ *and the parameter* x^* *of the best approximation lies in the ball* $B(x_0; r)$ *where* $r = \min\{d_0, \lambda_0/4K_0\}$. *We assume here that* $f \notin \mathscr{T}_{n,m}$.

Proof. The proof follows immediately from Theorem 1 once we note that

$$\epsilon_0 < \frac{\delta_0{}^2(\lambda_0)^{1/2}}{m^{1/2}2^{m+2}}$$

implies that the conclusions of Lemma 4 are valid and $\epsilon_0 \leqslant d_0/2K_4$ implies that $K_4\epsilon_0 < d_0$ so that hypothesis (3) of Theorem 1 is satisfied. Thus Theorem 1 applies and we are done. ∎

Remark 3. It is interesting to note the role of the parameter m in the above estimates. As m increases (that is, as the number of nonlinear parameters increases) the bounds decrease. This indicates that as the family $\mathscr{T}_{n,m}$ becomes more nonlinear the more "wavy" it is likely to be so that non-uniqueness is more likely close to the approximating family. We conjecture that as $m \to \infty$ (with n fixed or not) the least upper bound of the radius of unicity at $A(x)$ as x ranges over S will tend to zero.

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